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Instability conditions for a non-linear discrete system in the critical case of two unit roots☆

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ABSTRACT

A class of non-linear discrete second-order systems is considered in the critical case when two roots of the characteristic polynomial of the linearized system are equal to unity. Sufficient conditions for the instability of the equilibrium are obtained.

Interest in the question of the instability of non-linear discrete systems has been stimulated in recent years by the investigation of strange attractors, which sometimes arise in the simultaneous presence of dissipation and instability in the small.[1] Below, Lyapunous results on the instability of continuous steady-state non-linear systems in the critical case of two zero roots with a Jordan cell[2] are carried over to discrete systems for which the characteristic polynomial of the linearized system has two roots equal to unity. The reasoning is based on the construction of a special form of Lyapunov polynomial functions.

The system

$$\begin{aligned} x_{n+1} &= x_n + y_n[b_0 + b_1x_n + \ldots + b_{\alpha-2}x_n^{\alpha-2} + b(x_n, y_n)x_n^{\alpha-1}] + \kappa(x_n, y_n)y_n^2 \\ y_{n+1} &= y_n + ax_n^{\alpha} + p(x_n, y_n)x_n^{\alpha+1} + \\ &+ y_n[a_1x_n + \ldots + a_{\alpha-1}x_n^{\alpha-1} + c(x_n, y_n)x_n^{\alpha}] + \mu(x_n, y_n)y_n^2 \end{aligned} \quad (1)$$

is considered, where α is an integer, $\alpha \geq 2$, a, a_i and b_i are constants ($b_0 \neq 0$), and the functions $b(x_n, y_n)$, $\kappa(x_n, y_n)$, $p(x_n, y_n)$, $c(x_n, y_n)$ and $\mu(x_n, y_n)$ are bounded in the neighbourhood of the point $x_n = y_n = 0$.

Theorem. *The equilibrium state $x_n = y_n = 0$ of system (1) is unstable if either α is an even number or α is an odd number and $a > 0$.*

The following well known lemma (see Refs 3 and 4, for example) is used in the proof.

Lemma. *If, in the case of the system*

$$z_{n+1} = f(z_n), \quad z_n \in \mathbb{R}^n, \quad n = 0, 1, 2, \ldots$$

where $f(0) = 0$ in the neighbourhood of the point $z_n = 0$, a function of alternating sign $V(z_n)$ exists such that the function $W(z_n) = V(f(z_n)) - V(z_n)$ is of fixed sign, then the equilibrium state $z_n = 0$ is unstable.

Proof of the theorem. We will first consider the case of odd α and take the Lyapunov function in the form

$$V(x_n, y_n) = ax_ny_n + c_2y_n^2 + c_3x_n^3 + \ldots + c_{\alpha}x_n^{\alpha} \quad (2)$$

The constants $c_2, \ldots, c_{\alpha}$ will be chosen in such a way that this function satisfies the conditions of the lemma. We consider an increment in the Lyapunov function, calculated by virtue of system (1):

$$W(x_n, y_n) = V(x_{n+1}, y_{n+1}) - V(x_n, y_n)$$

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After the reduction of similar terms, we obtain

$$W(x_n, y_n) = ay_n^2 + a^2 x_n^{\alpha+1} + (\gamma_2 x_n^2 + \ldots + \gamma_\alpha x_n^\alpha) y_n + \nu_1(x_n, y_n) + \nu_2(x_n, y_n) \tag{3}$$

where

$$\lim \frac{\nu_1(x_n, y_n)}{y^2} = \lim \frac{\nu_2(x_n, y_n)}{|x|^{\alpha+1}} = 0 \text{ when } |x| + |y| \to 0 \tag{4}$$

The coefficients $\gamma_2, \ldots, \gamma_\alpha$ have the form

$$\gamma_2 = 3c_3 b_0 + aa_1, \quad g_3 = 4c_4 b_0 + 3c_3 b_1 + aa_2, \ldots$$

$$\ldots, \gamma_{\alpha-1} = \alpha c_\alpha b_0 + (\alpha - 1)c_{\alpha-1} b_1 + \ldots + 3c_3 b_{\alpha-3} + aa_{\alpha-2}$$

$$\gamma_\alpha = 2c_2 a + a^2 b_0 + 3c_3 b_{\alpha-2} + 4c_4 b_{\alpha-3} + \ldots + \alpha c_\alpha b_1$$

By successively solving the equations $\gamma_2 = 0$, $\gamma_\alpha = 0$, we find the coefficients $c_2, \ldots, c_\alpha$ and c_2.
As a result, we have

$$W(x_n, y_n) = ay_n^2 + a^2 x_n^{\alpha+1} + \nu_1(x_n, y_n) + \nu_2(x_n, y_n)$$

By virtue of the properties (4), the oddness of α and the positiveness of a, the function $W(x_n, y_n)$ is positive definite in the neighbourhood of zero. Since the function (2) is of variable sign, the instability of the equilibrium state follows from the lemma.

We now consider the case of even α and take the Lyapunov function of variable sign in the form

$$V(x_n, y_n) = y_n + ax_n y_n + c_2 x_n^2 + \ldots + c_\alpha x^\alpha$$

After the reduction of similar terms, we have

$$W(x_n, y_n) = \alpha x_n^\alpha + ay_n^2 + (\gamma_1 x_n + \ldots + \gamma_{\alpha-1} x_n^{\alpha-1}) y_n + \nu_3(x_n, y_n) + \nu_4(x_n, y_n)$$

where

$$\lim \frac{\nu_3(x_n, y_n)}{y^2} = \lim \frac{\nu_4(x_n, y_n)}{|\alpha|^\alpha} = 0 \text{ when } |x| + |y| \to 0 \tag{5}$$

As before, we define the coefficients $c_2, \ldots, c_\alpha$ in such a way that $\gamma_1 = 0$, $\gamma_{\alpha\text{-}1} = 0$. We obtain

$$W(x_n, y_n) = a(x_n^\alpha + y_n^\alpha) + \nu_3(x_n, y_n) + \nu_4(x_n, y_n) \tag{6}$$

By virtue of equality (5), the function (6) is of fixed sign and, consequently, according to the lemma instability occurs in the case of even α and $a > 0$.

Example. Consider two integrators which are spanned by crossed feedbacks which are implemented through amplitude-pulse modulators of the first kind[5,6] with instantaneous pulses are described by δ-functions

$$\dot{x} = u_1, \quad \dot{y} = u_2 \tag{7}$$

where

$$u_1(t) = \sum_{n=1}^{\infty} f_1(y_n)\delta(t - nT), \quad f_1(y_n) = y_n, \quad y_n = y(nT - 0)$$

$$u_2(t) = \sum_{n=1}^{\infty} f_2(x_n)\delta(t - nT), \quad f_2(x_n) = ax_n^\alpha, \quad x_n = x(nT - 0)$$

$T > 0$. Integrating system (7) from $nT - 0$ to $(n+1)T - 0$, we arrive at the equations

$$x_{n+1} = x_n + y_n, \quad y_{n+1} = y_n + ax_n^\alpha$$

According to the above theorem, the equilibrium state $x_n = y_n = 0$ is unstable when $\alpha \geq 2$ if α is either even or odd and $a > 0$.

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